Stability of solutions to Hamilton-Jacobi equations on closed domains arising in optimal control

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Hamilton-Jacobi equation

Consider a Hamiltonian $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that is convex in the last variable and the Hamilton-Jacobi equation

$$-v_t + H(t, x, -v_x) = 0, \quad v(T, \cdot) = \varphi(\cdot). \tag{1}$$

Let $H^*(t, x, \cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be the Fenchel conjugate of $H(t, x, \cdot)$ and consider the Calculus of Variations problem

$$v(t_0, x_0) = \inf\{\varphi(x(T)) + \int_{t_0}^{T} H^*(t, x(t), \dot{x}(t))dt \ :$$

$$x \in W^{1,1}([t_0, T), \ x(t_0) = x_0\}.$$

Faithful Representation of Hamiltonian

It is well known that under some assumptions on H, v is the unique viscosity solution of (1). However, H^* may have infinite values.
Question: can we associate to H mappings $f : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\ell : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ inheriting Lipschitz type regularity properties of H and such that

$$H(t, x, p) = \max_{u \in U}(\langle p, f(t, x, u)\rangle - \ell(t, x, u)), \tag{2}$$

where U is a compact subset of a finite dimensional space.

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where U is a compact subset of a finite dimensional space.

The answer is indeed positive.
That is H is equal to the Hamiltonian of a Bolza optimal control problem.

Value function of state constrained Bolza problem

Let K be a closed nonempty subset of $\mathbb{R}^n$.

$$V(t_0, x_0) = \inf \left\{ \varphi(x(T)) + \int_{t_0}^{T} \ell(t, x(t), u(t))dt \,\middle|\, (x, u) \in S_{[t_0,T]}(x_0) \right\}$$

Here, $S_{[t_0,T]}(x_0)$ denotes the set of all trajectory-control pairs of the control system under state constraint

$$\begin{cases} \dot{x}(s) = f(s, x(s), u(s)), \;\; u(s) \in U \; a.e. \; in \; [t_0, T] \\ x(t_0) = x_0 \\ x(s) \in K, \;\; \forall s \in [t_0, T] \end{cases}$$

Under appropriate assumptions, V is the unique solution to the Hamilton-Jacobi equation on the set $[0, T] \times K$.

Assumptions

(H1) $H(t, x, \cdot)$ is convex for any $(t, x) \in [0, T] \times \mathbb{R}^n$.

(H2) For any $R > 0$ there exists an integrable function $c_R \in L^1([0, T])$ such that for all $x, y \in B_R$ and $p \in \mathbb{R}^n$

$$|H(t, x, p) - H(t, y, p)| \leq c_R(t)(1 + |p|)|x - y|.$$

(H3) There exists an integrable function $c \in L^1([0, T])$ such that

$$|H(t, x, p) - H(t, x, q)| \leq c(t)(1 + |x|)|p - q|$$

for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and $p, q \in \mathbb{R}^n$.

(H4) $H^*(t,x,\cdot)$ is bounded on its domain for all $(t,x) \in [0,T] \times \mathbb{R}^n$.

(H5) For every $R > 0$ there exists $M_R > 0$ such that for all $(t,x) \in [0,T] \times B_R$ and $v \in dom(H^*(t,x,\cdot))$ we have

$$H^*(t,x,v) = \max_{p \in B(0,M_R)} (\langle v,p\rangle - H(t,x,p)).$$

(H6) For every $R > 0$ there exists an absolutely continuous $a_R \in W^{1,1}(0,T)$ such that for all $x \in B_R,\ p \in \mathbb{R}^n$ and $t,\ s \in [0,T]$

$$|H(t,x,p) - H(s,x,p)| \leq (1+|p|)|a_R(t) - a_R(s)|.$$

Regularity of dynamics and the cost function

Theorem

If (H1)-(H6) hold true, then $\exists f : [0, T] \times \mathbb{R}^n \times B \to \mathbb{R}^n$, *such that for* $\ell : [0, T] \times \mathbb{R}^n \times B \to \mathbb{R} \cup \{+\infty\}$ *defined by* $\ell(t, x, u) = H^*(t, x, f(t, x, u))$ *we have*

(A1) $H(t, x, p) = \max_{u \in B}(\langle p, f(t, x, u)\rangle - \ell(t, x, u)), \ \forall\ (t, x, p)$.

(A2) *For any* $R > 0$ *and for all* $t \in [0, T]$, $x, y \in B_R$, $u, v \in B$

$$\begin{cases} |f(t, x, u) - f(t, y, u)| \leq 10nc_R(t)|x - y| \\ |f(t, x, u) - f(t, x, v)| \leq 5n(1 + R)c(t)|u - v|. \end{cases}$$

(A3) $|f(t, x, u)| \leq c(t)(1 + |x|)$ *for all* $(t, x, u) \in [0, T] \times \mathbb{R}^n \times B$.

Regularity of dynamics and the cost function

Theorem

(A4) *ℓ takes finite values and for any $R > 0$, $t \in [0, T]$, $x, y \in B_R$, $u, v \in B$*

$$\begin{cases} |\ell(t,x,u) - \ell(t,y,u)| \leq (1 + 11nM_R)c_R(t)|x - y|, \\ |\ell(t,x,u) - \ell(t,x,v)| \leq 5nM_R(1 + R)c(t)|u - v|. \end{cases}$$

and

$$|\ell(t,x,u) - \ell(s,x,u)| \leq (1 + 11nM_R)|a_R(t) - a_R(s)|.$$

We associate to f, ℓ, φ the Bolza optimal control problem.

Stability of representations

If assumptions (H1)-(H6) hold true for continuous $H_i : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $i \geq 1$, then there exist $f_i : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $\ell_i : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ satisfying (A1)-(A4), which are standard in control theory.

Theorem

If H_i converge uniformly on compacts to some $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and satisfy $(H1) - (H6)$ with the same $M_R, c_R(\cdot), c(\cdot), a_R(\cdot)$, then the representations f_i, ℓ_i can be chosen to converge to some f, ℓ and satisfying (A1)-(A4) with the same $M_R, c_R(\cdot), c(\cdot), a_R(\cdot)$.

State constraints

We assume that the closed sets K and K_i are defined by the multiple inequality constraints

$$K \doteq \bigcap_{j=1}^{m} \{x : g^j(x) \leq 0\}$$

$$K_i \doteq \bigcap_{j=1}^{m} \{x : g_i^j(x) \leq 0\},$$

where $g_i^j : \mathbb{R}^n \to \mathbb{R}$ and $g^j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, .., m$, $i = 1, 2, \ldots$ are continuously differentiable functions satisfying

(G1)

i) For any $R > 0$ there exists $A_R > 0$ such that $|\nabla g_i^j(x)| \leq A_R$, for any $x \in RB$, ∇g_i^j is A_R-Lipschitz on RB.

ii) $\nabla g_i^j \to \nabla g^j$ uniformly on compacts and $g_i^j(0) \to g^j(0)$, when $i \to \infty$, for any $j = 1, \ldots, m$.

Inward pointing condition

For any $x \in \mathbb{R}^n$ denote by $I(x)$ the set of active indices at x for $g(\cdot) = (g^1(\cdot), ..., g^m(\cdot))$, i.e.

$$I(x) = \{j : g^j(x) = 0\}.$$

(IPC) For any $R > 0$ there exists $\rho_R > 0$ such that for every $x \in K \cap RB$ with $I(x) \neq \emptyset$ and every $s \in [0, T]$

$$\inf_{v \in dom(H^*(t,x,\cdot))} \max_{j \in I(x)} \langle \nabla g^j(x), v \rangle \leq -\rho_R.$$

Viscosity solutions of Hamilton-Jacobi equation

Definition

A continuous function $V : [0, T] \times K \to \mathbb{R}$ *is called a viscosity solution of Hamilton-Jacobi equation on the closed set* $[0, T] \times K$ *if* $V(T, \cdot) = \varphi(\cdot)$ *and*
i) for all $(t, x) \in [0, T] \times IntK$ *and all* $(p_t, p_x) \in \partial_- V(t, x)$

$$-p_t + H(t, x, -p_x) \geq 0$$

ii) for all $(t, x) \in [0, T] \times IntK$ *and all* $(p_t, p_x) \in \partial_+ V(t, x)$

$$-p_t + H(t, x, -p_x) \leq 0,$$

where $\partial_\pm V(t, x)$ *are the Frechet super/subdifferential of* V *at* (t, x).

Bilateral solution of Hamilton-Jacobi equation

Definition

$V : [0, T] \times K \to \mathbb{R}$ is called a bilateral solution of Hamilton-Jacobi equation on the closed set $[0, T] \times K$ if $V(T, \cdot) = \varphi(\cdot)$ and
i) for all $(t, x) \in [0, T] \times IntK$ and all $(p_t, p_x) \in \partial_- V(t, x)$

$$-p_t + H(t, x, -p_x) = 0$$

ii) for all $(t, x) \in [0, T] \times \partial K$ and all $(p_t, p_x) \in \partial_- V(t, x)$

$$-p_t + H(t, x, -p_x) \geq 0.$$

Uniqueness of solutions of Hamilton-Jacobi equation

Theorem

If assumptions (H1)-(H6) and (IPC) hold true. Then the associated value function of the Bolza optimal control problem is the unique bilaterial solution of the Hamilton-Jacobi equation on $[0, T] \times K$ *in the class of continuous functions.*

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Stability of Solutions to Hamilton-Jacobi equation

Theorem

Suppose that H_i satisfy (H1)-(H6) with the same $a_R(\cdot), c_R(\cdot)$, $c(\cdot)$, M_R and converge uniformly on compacts to a Hamiltonian H. Assume that (IPC) holds true. If φ_i converge to φ uniformly on compacts, then the unique solutions V_i to Hamilton-Jacobi equation with H_i, φ_i, K_i converge uniformly on compacts contained in the interior of K to the unique solution of Hamilton-Jacobi equation with H, φ, K.

Stability of Value functions: Corollary

Corollary

Suppose that H_i satisfy (H1)-(H6) with the same $a_R(\cdot)$, $c_R(\cdot)$, $c(\cdot)$, M_R and converge uniformly on compacts to a Hamiltonian H. Assume that (IPC) holds true. Then

$$Lim_{i\to\infty} graph V_i = graph V$$

and

$$Lim_{i\to\infty} epi V_i = epi V,$$

where the limit is taken in the Kuratowski sense.

Thank you for your attention.